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## LETTER TO THE EDITOR

# Geodesics and curvature of a group of diffeomorphisms and motion of an ideal fluid

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**Abstract.** Motion of an ideal fluid is represented as geodesics on the group of all volume-preserving diffeomorphisms. Explicit form of the geodesic equation is presented for the fluid flow on a three-torus. Riemannian connection, commutator and curvature tensor are given explicitly and applied to a couple of simple flows with the Beltrami property. It is found that the curvature is non-positive for the section of two  $ABC$  flows with different values of the constants ( $A$ ,  $B$  and  $C$ ). The present study is an extension of the Arnold's results in the two-dimensional case to three-dimensional fluid motions.

We are concerned with a method to connect the problem of hydrodynamics of an ideal (incompressible and inviscid) fluid with a problem of finding *geodesics* on the group of all volume-preserving diffeomorphisms. The fact that this group is the appropriate configuration space for the hydrodynamics of an ideal fluid was first remarked by Arnold [1]. Under the restriction of two-dimensional flows on the torus  $T^2 = \mathbb{R}^2 / (2\pi\mathbb{Z})^2$  ( $\mathbb{R}$ : space of real numbers;  $\mathbb{Z}$ : set of integers), explicit formulae for the commutator, inner product, Riemannian connection and geodesic equation are presented. These formulae allow us to calculate the Riemannian curvature tensors on the geodesics of any two-dimensional cross section. By the Jacobi equation in differential geometry, the stability of the geodesics is determined by the curvatures on them. By stability it is meant here kinematically how the difference of particle positions between two flows of different initial conditions develops with time. Negative curvature leads to enhanced growth of the deviation in initial conditions with time. (This problem is different from that of the dynamical stability of the velocity field.)

In this letter, we present an explicit form of the geodesic equation for the motion of an ideal fluid on a three-torus  $T^3$ . It is possible to determine an expression of the *Riemannian curvature* of the group of volume-preserving diffeomorphisms. It is interesting to find that the curvature turns out to be non-positive for the two-dimensional section consisting of a particular vector (velocity) field (i.e. a simple flow with the Beltrami property) and a general vector field.

Consider a flow of an ideal fluid of uniform density in an  $n$ -dimensional flow domain  $M$  (a bounded Riemannian manifold without boundary). The present analysis is aimed at the flows in three-dimensional space although the formulation is valid for  $n$ -dimensional flows. The manifold  $M$  is provided with the metric given by the inner product  $(X \cdot Y)$  and the covariant derivative  $\nabla_X Y$  (or Riemannian connection for this metric) at any point  $x \in M$  for any two vector fields  $X$  and  $Y$  on  $M$ . The covariant derivative is a differentiation of the vector field  $Y$  in the direction of  $X$ . When the velocity  $v_t(y)$  of fluid motion is given for  $y \in M$  and  $t \in \mathbb{R}$ , the particle motion  $y(t)$  is

described by the ordinary differential equation,  $dy/dt = v_t(y)$  with  $y(0) = x \in M$ , where  $x$  is the initial position of the particle. The solution is written in the form:  $y = g_t(x)$ , which describes a smooth curve in  $M$  starting from  $x$ , usually called *Lagrangian particle path*.

For each  $t$ , the mapping  $g_t: M \rightarrow M$  is an auto-diffeomorphism carrying every particle of the fluid in  $M$  from the place it was at time 0 to the place it is at time  $t$ , and in the other sense for the time parameter  $t$  in an open interval  $I \subset \mathbb{R}$ ,  $g_t$  describes time development of the configuration of particles, called a *flow*. In case that  $v_t(x)$  is divergence-free, the diffeomorphism is volume-preserving. The volume-preserving diffeomorphisms from  $M$  to itself form an infinite-dimensional group, which is denoted as  $\mathcal{D}_v(M)$  ( $\mathcal{D}_v$  for short), and the flow  $g_t$  is a *curve* in the group  $\mathcal{D}_v$ .

Here we make some remarks about the infinite Lie group  $\mathcal{D}_v$ . For two elements  $f$  and  $g$  in the group, the operation  $fg$  is defined by  $f \circ g$ , i.e.  $f \circ g(x) = f(g(x))$  for  $x \in M$ , where  $f \circ g$  is the right translation of  $f$  by an element  $g$  and unit element  $e$  is the identity map of  $M$ . The tangent space  $T_f \mathcal{D}_v$  is a set of all divergence-free vector fields along  $f$ , i.e. consisting of every vector field  $V$  such that  $V \circ f^{-1}$  is divergence-free. The group  $\mathcal{D}_v$  admits a Riemannian metric  $\langle \cdot, \cdot \rangle$  and Riemannian (Levi-Civita) connection  $\tilde{\nabla}$ .

The initial position  $x$  of all the particles is usually taken as the Lagrangian (particle) coordinates. The flow  $g_t$  starts at  $e$ , the identity map. The fluid velocity with respect to  $x$  at each time  $t$  is a tangent vector  $\dot{g}_t \in T_{g_t} \mathcal{D}_v$ . To get the *Eulerian* velocity field  $v_t(y)$  at a fixed point  $y$  in the space  $M$ , we must carry this vector to the tangent space at the identity  $e$ . This can be done by the right-translation  $v_t = \dot{g}_t \circ g_t^{-1}$ , or equivalently  $\dot{g}_t(x) = v_t(g_t(x))$ .

The metric for two tangent vectors  $X, Y \in T_f \mathcal{D}_v$  is defined by

$$\langle X, Y \rangle_f = \int (X(x) \cdot Y(x))|_{f(x)} dx \quad (1)$$

where  $dx$  is the (Riemannian) volume element on  $M$ , and  $Q|_f$  denotes the quantity  $Q$  evaluated at a point  $f$ . Due to the volume-preserving nature of the mapping, we have the *right invariant* property:  $\langle X \circ g, Y \circ g \rangle_{f \circ g} = \langle X, Y \rangle_f$  for  $f, g \in \mathcal{D}_v$ .

Concerning the connection  $\tilde{\nabla}_{\tilde{X}} \tilde{Y}$  ( $\tilde{X}, \tilde{Y}$  being two vector fields on  $\mathcal{D}_v$ ), first we consider right-invariant vector fields on  $\mathcal{D}_v$ , i.e.  $\tilde{X}^R(f) = X \circ f$ ,  $\tilde{Y}^R(f) = Y \circ f$  for  $f \in \mathcal{D}_v$  where  $X, Y \in T_e \mathcal{D}_v$  (divergence-free vector fields on  $M$ ). The connection  $\tilde{\nabla}$  determines a new right-invariant vector field  $\tilde{\nabla}_{\tilde{X}^R} \tilde{Y}^R$  on  $\mathcal{D}_v$  called covariant derivative of  $\tilde{Y}^R$  in the direction of  $\tilde{X}^R$  defined by

$$\tilde{\nabla}_{\tilde{X}^R} \tilde{Y}^R|_f = P[\nabla_X Y] \circ f \quad (2)$$

where  $\nabla$  is the Riemannian connection of  $M$ , introduced previously. The covariant derivative  $\nabla_X Y$  is not necessarily divergence-free for two divergence-free vector fields  $X, Y$  on  $M$ . The operation  $P[\cdot]$  denotes the projection to the divergence-free part on  $M$ . It is shown [2] that the operation (2) gives the right-invariant Riemannian connection on  $\mathcal{D}_v$  associated with the Riemannian metric (1).

Next we take a curve  $g_t$  in  $\mathcal{D}_v$  which satisfies  $\dot{g}_t = dg_t/dt = v_t \circ g_t = v_t(g_t)$ , that is, the flow  $g_t$  is generated by  $v_t$ . Let  $\tilde{X}_t$  be a vector field along  $g_t$  which is given by right-translation of a time-dependent vector field  $X_t$  on  $M$ :  $\tilde{X}_t = X_t \circ g_t$ . Then the connection determines a new vector field called the covariant derivative of  $\tilde{X}_t$  along  $g_t$ ,

$$\tilde{D}\tilde{X}_t/dt = (\partial X_t/\partial t + P[\nabla_{v_t} X_t]) \circ g_t. \quad (3)$$

Since  $\tilde{D}/dt$  is the differentiation with respect to the parameter  $t$ , we have to add the term  $\partial X_t/\partial t$ .

The operations  $\{\tilde{X}, \tilde{Y}\} \rightarrow \tilde{\nabla}_{\tilde{X}} \tilde{Y}$  and  $\tilde{X}_t \rightarrow \tilde{D}\tilde{X}_t/dt$  are characterized uniquely by the axioms in the Riemannian geometry [3, 4]. In particular, a vector field  $\tilde{X}_t$  along  $g_t$  is said to be *parallel* if the covariant derivative is identically zero. The curve  $g_t$  is a *geodesic* when its tangent vector  $\tilde{g}_t$  is parallel along itself (the curve  $g_t$ ). In  $\mathcal{D}_v$  endowed with the connection  $\tilde{\nabla}$ , the geodesics are defined by the equation

$$\tilde{D}\tilde{g}_t/dt = 0. \tag{4}$$

The principle of least action asserts that the motion of an ideal fluid is a geodesic with the (weak) Riemannian structure (1) and Riemannian connection (2). In fact, the action  $I$  is defined as

$$I = \int \langle \tilde{g}_t, \tilde{g}_t \rangle|_{g_t} dt = \int dt \int (v_t(x) \cdot v_t(x))|_x dx$$

and the variational problem leads to equation (4), [2].

Using the right invariant property of the metric  $\langle \cdot, \cdot \rangle$ , and identifying  $X_t$  as  $v_t$  in (3), we get the following equation:

$$\tilde{D}\tilde{g}_t/dt = (\partial v_t/\partial t + P[\nabla_{v_t} v_t]) \circ g_t = 0. \tag{5}$$

For the covariant derivative  $\nabla_{v_t} v_t$ , we have the orthogonal decomposition [2]

$$\nabla_{v_t} v_t|_e = \{P[\nabla_{v_t} v_t] - \text{grad } p\}|_e \tag{6}$$

where the function  $p$  is a smooth scalar function on  $M$ . Thus, the right-translation of equation (5) leads to

$$\partial v_t/\partial t + P[\nabla_{v_t} v_t] = 0 \tag{7}$$

or in view of (6),  $\partial v_t/\partial t + \nabla_{v_t} v_t = -\text{grad } p$ . This is a generalized expression of the Euler equation for an ideal fluid on Riemannian manifold  $M$  without boundary. It is shown that (7) holds in most general cases with boundary [2]. For a flat Cartesian space, the covariant derivative reduces to the form

$$\nabla_{v_t} v_t = (v \cdot \text{grad})v \tag{8}$$

for the velocity field  $v$  with cartesian components  $(v_i)$  at  $x = (x_i)$ . Thus in terms of the hydrodynamic notation for  $n = 3$ , we have recovered the Euler equation

$$\partial v/\partial t + (v \cdot \text{grad})v = -\text{grad } p \quad \text{div } v = 0 \tag{9}$$

where  $p$  is the pressure divided by the uniform fluid density.

In the Lie algebra of the group  $\mathcal{D}_v$ , the commutator  $[\cdot, \cdot]_*$  is defined for two divergence-free vector fields  $X, Y$  as

$$[X, Y]_* = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X. \tag{10}$$

The right-invariant property of the metric and connection allows us to make corresponding calculation at  $e \in \mathcal{D}_v$ . From now on we write  $\tilde{\nabla}$  instead of  $P[\nabla]$ .

Given divergence-free vector fields  $X, Y, Z$  and  $W$ , a new divergence-free vector field  $\tilde{R}(X, Y)Z$  called the curvature tensor is defined by

$$\tilde{R}(X, Y)Z = -\tilde{\nabla}_X \tilde{\nabla}_Y Z + \tilde{\nabla}_Y \tilde{\nabla}_X Z + \tilde{\nabla}_{[X, Y]_*} Z \tag{11}$$

and then the curvature  $\tilde{R}_{XYZW}$  is given by

$$\tilde{R}_{XYZW} = \langle \tilde{R}(X, Y)Z, W \rangle. \tag{12}$$

The sectional curvature for the section  $\sigma \subset T_e \mathcal{D}_0$  spanned by  $X$  and  $Y$  is

$$\tilde{K}(\sigma) = \langle \tilde{R}(X, Y)X, Y \rangle / (\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2)$$

where the denominator is the square of the area of parallelogram spanned by  $X$  and  $Y$ . As we mentioned before, the stability of a geodesic  $g_t$  is determined by the Jacobi field  $A_t$  along  $g_t$ . The norm  $|A_t| \equiv \sqrt{\langle A_t, A_t \rangle}$  denotes the evolution of distance between two geodesics  $g_t^0$  and  $g_t^s$  per unit variation of  $s$  which start from the same origin  $e$  with different initial tangent vectors  $X$  and  $X + sY$  respectively [5]:

$$|A_t| = |Y|t - \frac{1}{6}\tilde{K}(\sigma)t^3 + O(t^4). \tag{13}$$

Thus negative  $\tilde{K}(\sigma)$  means enhanced deviation of two geodesics in the section  $\sigma$  spanned by  $X$  and  $Y$ .

We will now investigate the three-dimensional fluid motion on the flat three-torus  $T^3 = \mathbb{R}^3 / (2\pi\mathbb{Z})^3$ , i.e.  $\mathbf{x} = \{(x_1, x_2, x_3); \text{mod } 2\pi\}$  for  $\mathbf{x} \in T^3$ , and the curvature of the group  $\mathcal{D}_0(T^3)$ . Note that this  $T^3$  is actually bounded manifold without boundary. The elements of the Lie algebra of the group  $\mathcal{D}_0(T^3)$  can be thought of as real periodic vector fields on  $T^3$  with the divergence-free property. Such periodic fields are represented by the real part of corresponding complex Fourier forms.

The Fourier base  $e^{ik \cdot \mathbf{x}}$  is denoted by  $\mathbf{e}_k$ , where  $\mathbf{k} = (k_i)$  for  $i = 1, 2, 3$ . We now complexify the Lie algebra, inner product, commutator and the Riemannian connection and curvature tensor, so that all these functions become linear (or multilinear) in the complex vector space of the complexified Lie algebra. The functions  $\mathbf{e}_k$  ( $\mathbf{k} \in \mathbb{Z}^3, \mathbf{k} \neq 0$ ) form a basis of this vector space. The velocity field is represented as

$$\mathbf{v}_i(\mathbf{x}) \equiv \mathbf{u}(t, \mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u}_{\mathbf{k}}(t) \mathbf{e}_{\mathbf{k}} \tag{14}$$

where  $\mathbf{u}(t, \mathbf{x})$  and  $\mathbf{u}_{\mathbf{k}}(t)$  have three components, the last also being written as  $u^i(\mathbf{k})$  occasionally ( $i = 1, 2, 3$ ). The Fourier components must satisfy the two properties

$$(\mathbf{k} \cdot \mathbf{u}_{\mathbf{k}}) = 0 \quad \mathbf{u}_{-\mathbf{k}} = \mathbf{u}_{\mathbf{k}}^* \tag{15}$$

to represent the solenoidal and real conditions, respectively, where the asterisk  $*$  denotes the complex conjugate. It should be noted that  $\mathbf{u}_{\mathbf{k}}$  has two independent polarizations consistent with the first condition.

Let us take four vector fields satisfying the conditions (15):  $\mathbf{u}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}, \mathbf{v}_l \mathbf{e}_l, \mathbf{w}_m \mathbf{e}_m, \mathbf{z}_n \mathbf{e}_n$ . Then we have the following expressions. From (1), the inner product is

$$\langle \mathbf{u}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}, \mathbf{v}_l \mathbf{e}_l \rangle = (2\pi)^3 (\mathbf{u}_{\mathbf{k}} \cdot \mathbf{v}_l) \delta_{0, \mathbf{k}+\mathbf{l}} \tag{16}$$

which is non-zero only for  $\mathbf{k} + \mathbf{l} = 0$ . The covariant derivative (8) is readily obtained. However, this does not satisfy the divergence-free property. The projected covariant derivative  $\tilde{\nabla}$  is

$$\tilde{\nabla}_{\mathbf{u}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}} \mathbf{v}_l \mathbf{e}_l = i(\mathbf{u}_{\mathbf{k}} \cdot \mathbf{l}) \frac{\mathbf{k} + \mathbf{l}}{|\mathbf{k} + \mathbf{l}|} \times \left( \mathbf{v}_l \times \frac{\mathbf{k} + \mathbf{l}}{|\mathbf{k} + \mathbf{l}|} \right) \mathbf{e}_{\mathbf{k}+\mathbf{l}}. \tag{17}$$

This is perhaps a new form. Equation (10) gives the commutator

$$[\mathbf{u}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}, \mathbf{v}_l \mathbf{e}_l]_* = i((\mathbf{u}_{\mathbf{k}} \cdot \mathbf{l}) \mathbf{v}_l - (\mathbf{v}_l \cdot \mathbf{k}) \mathbf{u}_{\mathbf{k}}) \mathbf{e}_{\mathbf{k}+\mathbf{l}}. \tag{18}$$

From the definitions (11) and (12), the curvature tensor is

$$\begin{aligned} \tilde{R}_{klmn} &= \langle \tilde{R}(\mathbf{u}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}, \mathbf{v}_l \mathbf{e}_l) \mathbf{w}_m \mathbf{e}_m, \mathbf{z}_n \mathbf{e}_n \rangle \\ &= (2\pi)^3 \left( \frac{(\mathbf{u}_{\mathbf{k}} \cdot \mathbf{m})(\mathbf{w}_m \cdot \mathbf{k})}{|\mathbf{k} + \mathbf{m}|} \frac{(\mathbf{v}_l \cdot \mathbf{n})(\mathbf{z}_n \cdot \mathbf{l})}{|\mathbf{l} + \mathbf{n}|} - \frac{(\mathbf{v}_l \cdot \mathbf{m})(\mathbf{w}_m \cdot \mathbf{l})}{|\mathbf{l} + \mathbf{m}|} \frac{(\mathbf{u}_{\mathbf{k}} \cdot \mathbf{n})(\mathbf{z}_n \cdot \mathbf{k})}{|\mathbf{n} + \mathbf{k}|} \right). \end{aligned}$$

The curvature  $\tilde{R}_{klmn}$  takes non-zero value only for  $k+l+m+n=0$  and non-zero  $k, l, m, n$ . (The terms with zero denominator should be deleted.)

The above expression of the curvature tensor is of the same form as that of Arnold [1] for the two-dimensional case. In the latter case the direction of the vector  $u_k$  is uniquely determined due to the condition (15), that is, for  $k=(k_1, k_2, 0)$  we have  $u_k=i(k_2, -k_1, 0)$  and similar expressions for the other vectors. Then it can be readily shown that our formulas just reduce to those of Arnold where our  $u_k e_k$  corresponds to his  $e_k$ .

An analysis for the distance of each particle convected by two different velocity fields of  $T^2$ , which is related to the Jacobi field (13), has been carried out by Hattori [6], where an  $L^2$ -distance between two mappings  $g_t^0$  and  $g_t^s$  (with small values of  $s$ ) defined by

$$d(g_t^0, g_t^s) = \left( \int_{T^2} |g_t^0(x) - g_t^s(x)|^2 dx dy \right)^{1/2} \tag{19}$$

(here  $x=(x, y)$ ) is found to evolve in the same way as (13) by means of the Taylor expansion. This analysis can be extended to  $T^3$  without difficulty.

The geodesic equation (7) reduces to

$$\frac{\partial u^m(k)}{\partial t} + i \sum_{p+q=k} \sum_{j,l} k_j \left( \delta_{ml} - \frac{k_m k_l}{k^2} \right) u^j(p) u^l(q) = 0$$

by using (17) where  $\delta_{ij}$  is Kronecker's delta. This is exactly identical to the Fourier representations of the Euler equation [7]. This equation is well known in turbulence theory.

As an application, we consider a flow with Beltrami property, that is, we assume that the velocity field  $U_p = u_p e_p + u_{-p} e_{-p}$ , i.e.  $2 \operatorname{Re}[u_p e^{ip \cdot x}]$ , satisfies the condition,  $\operatorname{curl} U_p = \lambda U_p$  for a parameter  $\lambda \in \mathbb{R}$ . This eigenvalue problem can be solved with  $\lambda^2 = |p|^2$ . It is readily shown that  $U_p$  is a steady-state solution. Let  $X = \sum v_i e_i$  be any velocity field satisfying (15). Using the above formulas, we obtain

$$\langle \tilde{R}(U_p, X) U_p, X \rangle = -(2\pi)^3 \sum_l \frac{|(u_p \cdot l)|^2}{|p+l|^2} \left| \frac{(u_p \cdot l)^2}{|(u_p \cdot l)|^2} (v_l \cdot p) - (v_{l+2p} \cdot p) \right|^2 \leq 0.$$

This non-positive property is considered to be a three-dimensional version of the Arnold's result for the curvature of the group  $\mathcal{D}_v(T^2)$  in any two-dimensional section containing the direction  $\xi$  represented by the stream function  $\frac{1}{2}(e_p + e_{-p})$ .

The above result can be extended to a two-mode Beltrami flow  $U_{p,q}$  which is defined by the linear combination  $U_p + U_q$  of two Beltrami flows  $U_p$  and  $U_q$ . Although it is not difficult to derive the curvature formula in the section of  $U_{p,q}$  and the general direction  $X$ , we show here another interesting example, that is the  $ABC$  flow [8] (a three-mode Beltrami flow) represented by

$$U_{ABC} = A[(i, 1, 0) e^{ix_3} + (-i, 1, 0) e^{-ix_3}] + B[(0, i, 1) e^{ix_1} + (0, -i, 1) e^{-ix_1}] \\ + C[(1, 0, i) e^{ix_2} + (1, 0, -i) e^{-ix_2}]$$

where  $A, B, C \in \mathbb{R}$ . Each term on the right-hand side satisfies the Beltrami property with  $\lambda = -1$ . We have another  $ABC$  flow  $U_{A'B'C'}$  for  $(A', B', C') \neq (A, B, C)$ . It is straightforward to show that

$$\langle \tilde{R}(U_{ABC}, U_{A'B'C'}) U_{ABC}, U_{A'B'C'} \rangle \\ = -2(2\pi)^3 \{ (AB' - BA')^2 + (BC' - CB')^2 + (CA' - AC')^2 \}$$

namely the curvature is negative. It is interesting to find that even in the case where both  $(A, B, C)$  and  $(A', B', C')$  are close and are not in *the domain of chaos* [8], particle motion by  $U_{A'B'C'}$  will not be predicted from the particle motion by  $U_{ABC}$  in the course of time. It is expected that the present method may be applied to the analysis of rate of growth of line-element or surface-element in various flow fields and turbulence.

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